Even Cycles

We would like to determine the extremal number of the even cycle C_{2k}. Let us first begin with a probabilistic construction[1] of a C_{2k}-free graph.

Theorem 1.

$$\mathrm{ex}(n, C_{2k}) \geq \frac{1}{8} n^{1+1/(2k-1)}.$$

That is, there exists an n-vertex C_{2k}-free graph with $\frac{1}{8}n^{1+1/(2k-1)}$ edges.

Proof. The construction is probabilistic with the deletion method. Consider the probability space of random graphs with edge probability

$$p = n^{1/(2k-1)-1}.$$

1: Finish the proof by counting the expected number of edges. Then an upper bound on the expected number of cycles of length $2k$. By considering their difference, conclude the theorem holds.

Solution: The expected number of edges is

$$\binom{n}{2} p \geq \frac{1}{4} n^{1+1/(2k-1)}.$$

The expected number of cycles C_{2k} is

$$\frac{n(n-1)(n-2)\cdots(n-2k+1)}{4k} p^{2k} \leq \frac{1}{4k} n^{2k} p^{2k} = \frac{1}{8} n^{1+1/(2k-1)}.$$

Let X be the random variable defined by the number of edges minus the number of copies of C_{2k}. Thus, by linearity of expectation we have

$$\mathbb{E}[X] \geq \frac{1}{8} n^{1+1/(2k-1)}.$$

By the probabilistic method, there exists a graph G with $X \geq \mathbb{E}[X]$. If we remove an edge from each C_{2k} in G we are left with an n-vertex C_{2k}-free graph with $e(G) \geq \frac{1}{8} n^{1+1/(2k-1)}$.

□

The upper-bound (which is widely believed to be correct) is from Bondy-Simonovits.

Theorem 2 (Bondy-Simonovits[2], 1974)**.** *For k natural, there is c (depending on k) such that*

$$\mathrm{ex}(n, C_{2k}) \leq c n^{1+1/k}.$$

The proof will require several intermediate lemmas. A **theta graph** is a cycle with an extra edge (**chord**). Let A and B be sets of vertices in a graph. An AB-**path** is a path that begins in A and ends in B. Recall that the **length** of a path is the number of edges (i.e., one less than the number of vertices).

Lemma 3. *Let H be a cycle with a chord and let A, B be a partition of vertices of H with at least one crossing edge (i.e., a non-trivial bipartition), then either H contains an AB-path of every length or A, B is a proper 2-coloring of H.*

[1] An explicit construction due to Lazebnik, Ustimenko, and Woldar gives an exponent of $1 + 2/(3k-3)$.

[2] The best-known constant $(80\sqrt{k \log k} + o(1))$ is due to Bukh and Jiang.

Proof. Consider A, B as a 2-coloring of H and for a vertex x, let $\chi(x)$ be the color class of x. Let the vertices of the cycle be $0, 1, 2, \ldots, n-1$ and the chord be the edge $0, r$.

Suppose ℓ is minimal such that there is no AB-path of length ℓ on the cycle, i.e, $\chi(x) = \chi(x+\ell)$ for each vertex x (and addition is modulo n).

2: Show that $\ell > 1$, consider $d = \gcd(\ell, n) = s\ell + tn$ for integers s and t and use it to conclude that $d = \ell$, $\ell | n$, and that for every value that is NOT a multiple of ℓ, there is an AB-path on the cycle.

Solution: Note that $\ell > 1$ (as otherwise the partition is trivial). Then if $d = \gcd(\ell, n)$, there exist integers[3] s and t such that $d = s\ell + tn$. This implies that there is no AB-path of length d, so $\ell = d$ by minimality. Therefore, $\ell | n$. By the same reasoning, we get that for every value that is NOT a multiple of ℓ, there is an AB-path on the cycle (and thus in H).

If $r < \ell$, then $k\ell + (r-1)$ is not a multiple of ℓ.

3: Therefore there is an AB-path of length $k\ell + (r-1)$ on the cycle. Use it (shifted) with the chord to find an AB-path of length ℓ.

Solution: By shifting the endpoints in increments of ℓ we can get that the chord is between two vertices on the path. Replacing r edges of the path with the chord gives an AB-path of length $k\ell$. That is, there is an AB-path using the chord of every length a multiple of ℓ.

Now let $r \geq \ell$ and assume that the following paths are not AB-paths for all $0 \leq j < \ell$

$$
\begin{gathered}
-j, -j+1, \ldots, 0, r, r-1, \ldots, r-\ell+j+1 \\
\ell-j, \ell-j-1, \ldots, 0, r, r+1, \ldots, r+j-1
\end{gathered}
$$

4: Investigate which (end points) need to have the same χ. Finally, conclude that $\ell = 2$ and finish the proof with it.

Solution: Therefore, $\chi(-j) = \chi(r-\ell+j+1)$ and $\chi(\ell-j) = \chi(r+j-1)$. Furthermore, observe that $\chi(-j) = \chi(\ell-j)$. This implies that $\chi(r+j+1) = \chi(r-\ell+j-1) = \chi(r+j-1)$ This holds for all $0 \leq j < \ell$ so we can extend this pattern to the rest of the cycle, i.e., $\ell = 2$. Therefore the cycle is even. Furthermore, if the endpoints of the chord are in different classes then we have a proper 2-coloring of H and if the endpoints of the chord are in the same class we can find AB-paths of all lengths.

[3] By Bézout's identity: if ℓ and n non-zero, then there exists integers s and t such that $\gcd(\ell, n) = s\ell + tn$.

Finally, we are left with Now let $r \geq \ell$, and there is some $0 \leq j < \ell$ such that one of the above paths is an AB-path. WLOG assume

$$-j, -j+1, \ldots, 0, r, r-1, \ldots, r-\ell+j+1$$

is an AB-path of length ℓ.

5: Show existence of AB-path for every $k\ell$. What happens towards the end? Hint: extend the given path.

Solution: Then, extend the endpoints of this path in increments of ℓ until further extension would reuse a vertex of the path. So, one endpoint is among the vertices $1, 2, \ldots, \ell$ and the other is among $r+1, r+2, \ldots, r+\ell$. Therefore, of the $n+1$ edges in H, there are at most $2(\ell-1)$ not used by this AB-path, i.e., it is of length $\geq n+1-2(\ell-1) = n-2\ell+1$. However, because $\ell | n$ we must have that the AB-path is of length $n-\ell$. From this AB-path of length $n-\ell$ we can get every length $k\ell$ AB-path.

□

Because every graph with dn edges (i.e., average degree $2d$) contains a subgraph of minimum degree at least d, Bondy-Simonovits follows from

Theorem 4. *If G is an n-vertex graph with minimum degree greater than $2kn^{1/k}$, then G contains a C_{2k}.*

Proof. Suppose (to the contrary) that G is an n-vertex C_{2k}-free graph with minimum degree greater than $2kn^{1/k}$. Recall that the **distance** between two vertices is the length (number of edges) of the shortest path between them. Let v be an arbitrary vertex of G and for $1 \leq i \leq k$, define V_i as the set of vertices at distance i from v.

Claim 5. *Neither the graphs on V_i nor the bipartite graphs on V_i, V_{i+1} contain a bipartite theta graph on at least $2k$ vertices.*

Proof. Suppose, to the contrary, that the graph on V_i contains a bipartite theta graph on at least $2k$ vertices. Denote the theta graph by H and let X, Y be a bipartition with X minimal.

First we perform a breadth-first search of G starting at v. That is, we assign labels $1, 2, \ldots, n$ in order to the vertices of G as follows. First label v, then label the neighbors of v (in any order). From this point on, choose the vertex with lowest label that has some unlabeled neighbors and label these neighbors (in any order). This procedure produces a natural spanning tree of G. In particular, for every vertex we keep the edge connecting it to its neighbor with lowest label. This is the breadth-first spanning tree T of G.

Let x be such that all vertices in X are descendants (in the spanning tree T) of x and the distance ℓ between x and V_i is minimal (such a x exists as v certainly has all of X as descendants). Let y be a child of x such that there is an element of X among the descendants of y (if y itself is an element of X, then the proof proceeds similarly). Let A be the descendants of y in X and $B = (X \cup Y) - A$, i.e, A, B is a partition of H.

6: Finish the proof by finding a cycle of length $2k$.

Solution: If $X - A$ is empty, then y would be the vertex of maximum distance from v as defined above. Thus, $X - A \neq \emptyset$ and by minimality of X we must have an edge in B. Therefore, by the lemma from the previous lecture the bipartition A, B of H contains an AB-path of every length. In particular, there is an AB-path of length $2k - 2\ell$. Let $a \in A$ and $b \in B$ be the endpoints. Clearly, $a \in X$ and as the path is of even length, we have $b \in X$. That is b is not a descendant of y. Thus, there are edge-disjoint paths (of length ℓ) from a to x and b to x. These three paths together form a cycle of length $2k$; a contradiction.

The case when V_i, V_{i+1} contains a bipartite theta graph on at least $2k$ vertices is similar; put $X = H \cap V_i$. □

The next claim is left as an exercise.

Claim 6. *If H is a bipartite graph with minimum degree $\delta(H) \geq 3$, then H contains a (bipartite) theta graph on at least $2\delta(H)$ vertices.*

7: Bound the number of edges inside each V_i and the number of edges between V_i and V_{i+1}. Show that by the minimum degree condition, $|V_{i+1}| > n^{1/k}|V_i|$ for all i and then finish the proof by showing $|V_k|$ would be too large. Hint: Use that $n^{1/k} > c \cdot k$ for sufficiently large n.

Solution: Now let us establish bounds on the number of edges $e(V_i, V_{i+1})$ between V_i and V_{i+1}. By the two claims, no subgraph of $G[V_i, V_{i+1}]$ has minimum degree k, thus $G[V_i, V_{i+1}]$ as average degree at most $2k-2$. That is,

$$e(V_i, V_{i+1}) < (2k-2)|V_{i+1}|.$$

Now observe that if V_i contains at least $2k|V_i|$ edges, then it contains a bipartite subgraph with $k|V_i|$ edges (Exercise **??**). This subgraph has average degree $2k$, so it contains a (bipartite) subgraph of of minimum degree k and the two claims give a contradiction. Thus, the number of edges contained in V_i is less than $2k|V_i|$. On the other hand, each vertex in V_i has degree greater than $2kn^{1/k}$ in G and only has neighbors in V_{i-1}, V_i, and V_{i+1}. Thus,

$$e(V_i, V_{i+1}) > 2kn^{1/k}|V_i| - 2e(V_i) - e(V_{i-1}, V_i) \geq 2kn^{1/k}|V_i| - 4k|V_i| - 2k|V_i|.$$

Clearly, for n large enough we have $n^{1/k} > 6k$, thus

$$e(V_i, V_{i+1}) > (2k-1)n^{1/k}|V_i|.$$

Combining the two estimates for $e(V_i, V_{i+1})$ gives

$$|V_{i+1}| > n^{1/k}|V_i|.$$

Repeatedly applying this bound and using the fact that $|V_0| = |\{x\}| = 1$ gives

$$|V_k| > \left(n^{1/k}\right)^k = n$$

an obvious contradiction.

□

Open problem: Construct an n-vertex C_8-free graph with $\Omega(n^{5/4})$ many edges.